Infinite Products

It's not obvious what the definition of convergence of infinite products should be. For certain reasons, we don't want a product Πp_n to converge to 0. This would be analogous to allowing a sum to converge to $\pm\infty$. So we require that the limit be non-zero. It is convenient to require that the terms $p_n \neq 0$. We can form a partial product $P_m = \Pi_1^m p_n$ and and introduce a_n such that $p_n = 1 + a_n$. Then we have the

Definition 1. $\Pi_1^\infty p_n$ ***converges*** *if* $\lim_{n\to\infty} P_n = P \neq 0$.

Notice this implies that $p_n \neq 0$. Also this implies that $p_{n+1} = \frac{P_{n+1}}{P_n} \to 1$ and hence that $a_n \to 0$. The proof of the following theorem takes some care. It is not always done correctly. I'm taking this proof from Alhfors [1].

Theorem 1. $\Pi_1^\infty p_n$ *converges if and only if* $\sum_1^\infty \log(1 + a_n)$ *converges, where* $\log(1 + a_n)$ *is the principal value of the logarithm. It is not necessarily true that* $\log(\Pi_1^\infty)$ *is the equal to* $\sum_1^\infty \log(1 + a_n) = S$.

Proof. If $\sum_1^\infty \log(1 + a_n)$ converges and is equal to S, then it follows, by exponentiating, that $e^S = e^{(\sum_{n\to\infty} \log(1+a_n))} = \lim_{n\to\infty} P_n$. This is the easy part.

Now assume that $\Pi_1^\infty(1 + a_n)$ converges to P. Let $S_n = \sum_{m=1}^n \log(1 + a_m)$. Then $\log(\frac{P_n}{P}) \to 0$ and $\log(1 + a_n) \to 0$. There is the following relation between the principal values of the logarithms,

$$\log(\frac{P_{n+1}}{P}) = S_{n+1} - \log(P) + 2\pi i q_{n+1} \quad (1)$$

$$\log(\frac{P_n}{P}) = S_n - \log(P) + 2\pi i q_n, \quad (2)$$

where $q_n, q_{n+1} \in \mathbb{Z}$. Subtract equation (2) from (1) to get

$$\log(\frac{P_{n+1}}{P}) - \log(\frac{P_n}{P}) = \log(1 + a_{n+1}) + 2\pi i(q_{n+1} - q_n).$$

This equation implies that $(q_{n+1} - q_n) \to 0$ and since the the q_n are integers $q_n = q_{n+1} = q$ for all $n > N$. From equation (2), we conclude that S_n converges to $\log(P) - 2\pi i q$. In other words

$$\sum_1^\infty \log(1 + a_n) = \log(\Pi(1 + a_n)) - 2\pi i q.$$

□

References

[1] Ahlfors, Lars; Complex Analysis, McGraw-Hill, 1979.